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# THE MATHEMATICS TEACHER

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## THE TEACHING OF VERBAL PROBLEMS.

BY E. R. BRESLICH.

Verbal problems in algebra are usually regarded by the pupil as the most difficult part of the course. This difficulty arises out of a number of causes. The ideas involved are sometimes remote from the pupil's experience and are therefore of little interest to him.

Since various types of verbal problems, each calling for a special method of attack, are usually given in the same set of exercises, the difficulty is increased. If, as it frequently happens, these problems are given at the end of the chapter, there is a tendency on the part of the teacher toward passing over this work hurriedly, especially when more than the available time has been spent on the early part.

The information being given in words, and not in symbols, is not as easily classified under familiar mathematical categories as much of the formal work of algebra. For example, let it be required to write the expression

$$\sqrt{\frac{36a^3}{34b} \cdot \frac{32c^{-2}}{24a} \cdot \frac{9b}{8c^4}}$$

in the simplest form. Although at the first glance, the problem as a whole might impress the pupil as difficult, he will readily see upon closer analysis that the symbols suggest a number of things he may do. He may multiply the fractions under the

radical sign, or reduce them to the simplest form. He may square some of the numbers as indicated, or extract the square roots of some of the factors. The negative exponent might suggest a law by which he may change the power into one with a positive exponent. It will be to his advantage to decide which of these operations better be performed first to get the simplest solution of the problem, but any one of them will enable him to get a start. There is nothing in the statement of a verbal problem to suggest to the pupil a method of procedure. The detailed analysis which must be made before the pupil can derive the equation is difficult to make, and takes time and patience. It is a type of activity which does not appeal to many pupils of high-school age as much as the manipulation involved in formal work.

However, the principal reason for the apparent difficulty in the verbal problems is the lack of a good technique for attacking and solving them. This is usually not as definitely worked out as for the formal work. Without such a technique the pupil loses much time while he is wondering what he should do, and when he finally succeeds in working one problem he does not necessarily know how to apply his experience even to another problem of the same type.

The fact that pupils spend a large amount of time trying to find a method for solving a verbal problem has sometimes been interpreted to mean that verbal problems require more real thinking than formal work. The trouble with the abstract work is that it is nearly all technique but lacks application. It therefore involves no functional learning. On the other hand, problem work gives much opportunity for the development of mathematical adaptability, but frequently lacks the necessary technique. With a technique directing his thinking similar to that of the formal work the solution of verbal problems may be made a matter as simple, or even more simple than most of the formal work.

When it is considered that in the ordinary textbook on mathematics about one third of the space is given to verbal problems, the need for a good technique becomes still more apparent. Some writers recognize the importance of functional learning involved in problem work have advocated a further increase in

the number of verbal problems, and it is possible that future books may include an increasingly larger amount of this type of work. The technique of teaching verbal problems should therefore be made an object of careful study on the part of every teacher of mathematics.

The pupil knows, of course, that in every verbal problem there is an unknown number, the value of which is to be determined by solving an equation. The difficulty usually lies in deriving this equation. The solution of the equation is then mostly a simple matter.

The pupil therefore needs some general directions showing him how to begin in every case, and some additional particular suggestions which he may apply to particular types of verbal problems. These directions should be so definite that the teacher should get immediate action from every pupil the moment a problem is assigned to him. Furthermore, whenever a problem is explained to the class, the suggestions should help to make the explanation so clear that no pupil will fail to profit by it.

However, it must be mentioned that there is a real danger in using too much of this type of work. The directions given to the pupil should be such that he may understand the method clearly and so that the teacher knows that he sees the process. But they should not be emphasized so much that the pupil acquires the habit of having all his adaptations made for him. There must be a great deal of problem work for which the pupil must develop his own technique. This should be done in the class room with frequent consultations with the teacher. Unless the pupil is allowed ultimately to make adaptations alone without further outside help he will become helpless and will not develop adaptability.

On the other hand, just as in the formal work certain types of situations occur so frequently that the teacher is justified in deliberately having pupils memorize certain formulas; he may set up a framework for the solution of certain verbal problems and drill the pupils in its use, thereby relieving them of a good deal of mechanical detail.

We shall first formulate some suggestions which will help the pupil with a particular problem.

*In many verbal problems the equation is obtained by translating the statement word for word into algebraic language. The following is a typical example.*

The ratio of two numbers is  $a$ . If the first is increased by  $m$  and the second decreased by  $n$  the ratio of the results is  $3/2$ . What are the numbers?

Using one unknown, the solution is as follows.

Let  $ax$  be one number, and  $bx$  the other.

Then  $ax + m$  is the first number increased by  $m$  and  $bx - n$  is the second number decreased by  $n$ .

Translating into symbols the statement "the ratio of the result  $s$  is  $3/2$ ," we have the equation

$$\frac{ax + m}{bx - n} = \frac{3}{2},$$

which may be solved for  $x$ . Then  $ax$  and  $bx$  are the required numbers.

Using two unknowns, we proceed as follows.

Let  $x$  be one number and  $y$  the other.

Translating the statement of the problem, we have

$$\frac{x}{y} = \frac{a}{b} \quad \text{and} \quad \frac{x + m}{y - n} = \frac{3}{2},$$

The values of  $x$  and  $y$  are then obtained by solving this system of equations simultaneously.

The pupil's attention should be called to the useful device of representing in terms of one letter two or more numbers whose ratios are given. For example, two numbers in the ratio  $4/5$  may be denoted by  $4x$  and  $5x$ . Three numbers in the ratio  $4:5:6$  may be denoted by  $4x$ ,  $5x$ , and  $6x$ , respectively.

*The facts involved in verbal problems are usually related to each other.* If the pupil is made familiar with these relations, he will be able to derive some of the facts from the others. The motion problem is the most common of this type. Here we have the relations  $d = rt$ ,  $r = d/t$ ,  $t = d/r$ . By means of those relations the pupil can obtain any one of the three elements provided he knows the other two. He should therefore know the necessity and advantage of taking this step. Clearness can be attained by arranging the work in tabular form. To illustrate

the method, let us consider in detail the solution of a problem of this type.

A man travels to a certain place at the rate of 18 miles an hour. He returns on a road  $6\frac{1}{2}$  miles shorter, and makes the trip in 29 minutes less time, although travelling only 17 miles per hour. How long is each road?

The pupil may arrange the facts involved in a double column table, one for each road, as follows:

Elements.	Long Road.	Short Road.
$d$ .....	$x$ miles	$x - 6\frac{1}{2}$ miles
$r$ .....	18 miles per hour	17 miles per hour
$t$ .....	$x/18$ hours	$\frac{x - 6\frac{1}{2}}{17}$ hours

The first element  $d$  is denoted by  $x$  because it is the unknown in the problem. The second element  $r$  is given. The third is derived from the first two by means of the equation  $t = d/r$ . Not before he has completed this table, should the pupil try to state the equation. The 20 minute difference in the time, which has not been used in the table gives the equation. Since the hour is the time unit, the 20 minutes must be changed to a fraction of an hour. The equation is therefore

$$\frac{x}{18} = \frac{x - 6\frac{1}{2}}{17} + \frac{1}{3}, \quad \text{not} \quad \frac{x}{18} = \frac{x - 6\frac{1}{2}}{17} + 20.$$

A study of the difficulties involved in this type of problem has shown that pupils are usually not successful in deriving the equation if they fail to state first all three of the facts in the table, and that many fail to recognize the necessity of using the same unit of time throughout the equation. These two points should therefore be emphasized in the teaching of verbal problems.

This technique may be used in a surprisingly large number of verbal problems found in textbooks on algebra. The following table is a selection of the best known types whose solution may be simplified by the method just described.

The scheme of tabulating the elements involved in a problem may be used with advantage in some problems in which there is no equation expressing directly a relation between the elements. The so-called work problems and loss of weight problems are

Type of Problem.	Elements.		Relation
Earning money .....	Number of days,	$n$	$p = rn$
	Pay per day,	$r$	
	Total pay,	$p$	
Selling and buying.....	Number sold,	$n$	$p = rn$
	Price of each,	$r$	
	Total price,	$p$	
Gain and loss .....	Purchase price,	$p$	$s = p + g$
	Gain,	$g$	
	Selling price,	$s$	
Interest, discount .....	Principal,	$p$	$i = (r.p.t.)/100$
	Rate,	$r/100$	
	Time,	$t$	
	Interest,	$i$	
Leverage .....	Arm,	$a$	$l = af$
	Force,	$f$	
	Leverage,	$l$	
Carriage wheel .....	Distance,	$d$	$d = nc$
	Circumference,	$c$	
	Number of revolutions,	$n$	

typical examples. In the first the elements are the number of days, the amount done in one day, and the amount done in a given number of days.

ILLUSTRATIVE PROBLEMS.

1. *A* and *B* together can do a piece of work in 12 days. After *A* has worked alone for 5 days, *B* finishes the work in 26 days. In what time can each alone do the work?

The following table gives the facts.

	<i>A.</i>	<i>B.</i>
Number of days it takes to do the work alone.....	$x$	$y$
Part of whole done in one day.....	$1/x$	$1/y$
Amount done in 12 days.....	$12/x$	$12/y$
Amount done in 5 days.....	$5/x$	....
Amount done in 26 days.....	...	$26/y$

The equations are obtained by expressing in several ways the whole piece of work. Thus we have

$$\frac{12}{x} + \frac{12}{y} = 1 \quad \text{and} \quad \frac{5}{x} + \frac{26}{y} = 1.$$

2. If 19 lb. of gold and 10 lb. of silver each lose 1 lb. when weighed in water, how much gold and how much silver is contained in a mass of gold and silver that weighs 80 lb. in air and  $72\frac{1}{2}$  lb. in water?

	Gold.	Silver.
Amount in mass .....	$x$	$80 - x$
Loss of weight of 1 lb.....	$1/19$	$1/10$
Loss of weight of several lb.....	$x/19$	$(80 - x)/10$

The equation expresses the sum of the partial losses as equal to the total loss:

$$\frac{x}{19} + \frac{80 - x}{10} = 7.5.$$

The following examples show in detail arrangements of the steps involved in the solution of a number of typical verbal problems not mentioned above. These arrangements have been found to be helpful to the teacher in presenting the problem and to the pupil in finding the solution.

1. A man has two sons, one six years older than the other. After two years the father's age will be twice the combined ages of his sons, and six years ago his age was four times their combined ages. How old is each?

Let  $x$  be the number of years in the father's age.

Let  $y$  be the number of years in the age of the younger son.

Then  $y + 6$  is the number of years in the age of the older son.

After two years  $\left\{ \begin{array}{l} \text{the father's age is } x + 2 \text{ years,} \\ \text{the younger son's age is } y + 2 \text{ years,} \\ \text{the older son's age is } y + 8 \text{ years.} \end{array} \right.$

$$\text{Hence } x + 2 = 2(y + 2 + y + 8).$$

Six years ago  $\left\{ \begin{array}{l} \text{the father's age was } x - 6 \text{ years,} \\ \text{the younger son's age } y - 6 \text{ years,} \\ \text{the older son's age } y \text{ years.} \end{array} \right.$

$$\text{Hence } x - 6 = 4(y - 6 + Y).$$

2. The tens digit of a number of two digits is 6. If the order of the digits be reversed, the resulting number is 27 greater than the original number. Find the number.

Before taking up the solution of this problem the teacher should tell the class something about the principle of position in arithmetic. It may be shown that an arithmetical number, such as 256, is a brief way of writing  $200 + 50 + 6$ , or  $2.100 + 5.10 + 6$ . A number of three digits in algebra cannot be expressed in as simple a form as 256, since it is agreed that  $x y z$  should mean a product of the three factors  $x$ ,  $y$ , and  $z$ . Hence, the longer form must be used, giving the number  $100x + 10y + z$ . If the digits are reversed the resulting number will be  $100z + 10y + x$ .

Although the unknown called for in the problem is the required number of two digits, it is convenient to denote the unit digit by  $x$ .

Then the number is  $6.10 + x$ .

The number resulting by reversing the digits is  $x.10 + 6$ .

Hence  $10x + 6 = 27 + (60 + x)$  is the equation from which the value of  $x$  is determined. Having found the value of  $x$  it is a sample matter to find the number  $60 + x$ .

3. How much water must be added to 12 gallons of a 25 per cent. solution of alcohol and water to reduce it to a 10 per cent. solution?

Let  $x$  be the number of gallons of water to be added. Then the facts may be arranged as in the following table.

	Old Solution.	New Solution
The amount of alcohol		
and water .....	12 gallons	$(x + 12)$ gallons
The amount of alcohol...	$25/100$ 12 gallons	$10/100$ $(x + 12)$ gallons

Since the amount of alcohol has not been changed in diluting the solution, we have the equation

$$\frac{25}{100}(12) = \frac{10}{100}(x + 12).$$

4. The time of revolution of Venus about the sun is about  $7\frac{1}{2}$  months, and that of Mercury 3 months. How many months after Mercury is in the line between Venus and the Sun will it next be in the same relative position?

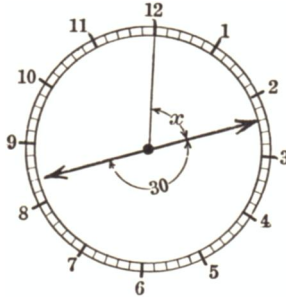
	Venus.	Mercury.
The required number of		
months .....	$x$	$x$
Time of revolution.....	$7\frac{1}{2}$	3 months
Part of orbit passed over in		
1 month .....	$1/7.5$ of a revolution	$1/3$ of a revolution
Part of orbit passed over in		
$x$ months .....	$x/7.5$ of a revolution	$x/3$ of a revolution

Since Mercury makes one revolution more than Venus, we have the equation

$$\frac{x}{3} = 1 + \frac{x}{7.5}.$$

5. At what time between eight and nine o'clock are the hands of a clock opposite to each other?

Since the time is given by the number of minutes after eight o'clock, let  $x$  be the number of minute spaces passed over by the minute hand since eight o'clock.



As the minute hand passes over 60 minute spaces, the hour hand passes over five, *i. e.*, over  $1/12$  as many as the minute hand.

Therefore  $x/12$  is the number of minute spaces passed over by the hour hand since eight o'clock. The equation may be obtained from a drawing. Thus, we may express the distance from 12 to the hour hand in minute spaces. This gives the equation

$$x + 30 = 40 + \frac{x}{12}.$$

We may also express the distance from 12 to the minute hand, giving the equation.

$$x = 10 + \frac{x}{12}.$$

In the last equation use is made of the fact that the distance from 2 to the minute hand is the same as the distance from 8 to the hour hand, which is based upon the equality of the two opposite angles at the center.

6. *Geometric Problems.*—The relative number of verbal problems of geometric content is gradually increasing not only in the newer textbooks, but in the revisions of the older ones. The equation in this type of problems usually expresses some geometric fact, such as the sum of the angles of a triangle is 180 degrees, the area of a circle is  $\pi r^2$ , the volume of a rectangular

box is the product of the three dimensions, and the square on the hypotenuse is equal to the sum of the squares of the sides.

As a rule these problems are not very difficult because the equation is easily derived by translating the geometric principle into algebraic symbols.

The following simple directions apply to all verbal problems and aim to help the pupil to get an immediate start.

1. *Read the problem carefully.* The problem may be read silently by all pupils, or aloud by one pupil. The purpose of this is not primarily to teach pupils to read accurately. We know that some have difficulty in reading. It may be possible and desirable to test pupils from time to time throughout the course to see whether or not they develop in reading ability, and to give them training in reading as applied to the mathematical field. The suggestion that the pupil should read the problem carefully is here given with the purpose of breaking pupils of the habit of attempting to solve a problem without first getting clear notions regarding the facts involved. This first reading is to get the pupil started and to give him a general idea of the content. Pupils should not try to study particular parts of the problem at this time.

2. *By further reading find what the problem asks for.* This will help the pupil to determine the unknown number, or numbers.

3. *Let  $x$  denote the unknown number.* Any other letter, of course, may be used as well. Emphasis must be placed on a definite statement in writing. Thus, if the rate is the unknown, the pupil should write: "let  $r$  be the number of miles per hour," not "let  $r$  be the rate." If the price is called for, the statement should be: "let  $n$  be the number of cents," not "let  $n$  be the price." In this manner the pupil will cultivate the desirable habit of always defining the letter before using it in an equation. Although, as a rule, the pupil should denote by a letter the unknown called for in the problem, there are exceptions. The solution in some cases may be simplified by choosing some other number as the unknown. The following problem illustrates this: "The denominator of a fraction is 5 greater than the numerator. If 1 is added to the numerator, the value of the resulting fraction is  $\frac{1}{2}$ ." The problem contains three unknown

numbers, the numerator, the denominator, and the fraction. Denoting the numerator by  $n$ , the denominator is  $n + 5$ , the fraction is  $n/n + 5$ , and the equation takes the simple form

$$\frac{n}{n + 5} = \frac{1}{2}.$$

4. *Read the problem again, one sentence at a time, and express the various facts involved in terms of the unknown literal number.* For example, let the first sentence read: "A sum of \$5,330 is to be divided into two parts." The two parts may be denoted by two letters such as  $x$  and  $y$ . Then the first sentence will then read  $x + y = 5,330$ .

If only one unknown is to be used, the pupil should write these facts as follows.

Let  $x$  be the number of dollars in one part.

Then 5,330 —  $x$  is the number of dollars in the other.

If the teacher insists upon these four steps in the solution of all verbal problems he will train the pupil continuously in expressing in terms of algebraic symbols facts given in the form of verbal statements. Moreover, every pupil will at least get a start which usually results in a successful solution of the problem.

Experiments have shown that efficiency in solving verbal problems is attained more rapidly by individual effort of the pupil than by explanation of the solutions of a large number of problems. A new type of problem should first be presented by the teacher, emphasizing the technique as he works out the solution. A model solution should then be written on the blackboard. Next individual assignments should be made and help should then be given only after the pupil has made serious effort. A problem should be explained to the class only after every pupil has done considerable work on the problem. Explaining problems which the majority of pupils in the class have never tried to solve involves a relatively great waste of time.

Since the solution of the equations obtained from verbal problems usually offers no difficulty, many of the verbal problems may be worked only to the point of stating the equation, or equations. This will give a large amount of training in translating verbal statements into algebraic symbols.

Where solutions are worked out completely, they should frequently be checked in the original problem, because the roots of the equation do not necessarily satisfy the conditions of the problem: The following two examples illustrate this.

1. Find the length of the altitude of an equilateral triangle in terms of the side.

Denoting the length of the altitude by  $h$  and the length of the side by  $a$ , we have the relation

$$a^2 = h^2 + \frac{a^2}{4},$$

from which we find that

$$h = \pm \frac{a}{2} \sqrt{3}.$$

Evidently only  $h = +2/2\sqrt{3}$  satisfies the condition of the problem.

2. According to the law of falling bodies, the time  $t$ , in which a ball thrown upward with a velocity of 100 ft. per second reaches a height of 144 ft., is given by the equation  $144 = 100t - 16t^2$ . Find the value of  $t$ .

The values of  $t$  satisfying this equation are  $2\frac{1}{4}$  and 4. Both of these clearly satisfy also the condition of the given problem.

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